By applying the von Neumann stability analysis it is seen that the stability conditions of (6.72) with constant coefficients are given by

$$\text{(i)} \quad \sigma < \frac{1}{4}, \tau \geqslant \frac{1}{4}, p > 0$$

$$\text{(ii)} \quad \sigma \leqslant \frac{1}{4}, \tau < \frac{1}{4}, 0 < \sqrt{a}p \leqslant \sqrt{\frac{1-4\sigma}{1-4\tau}} \tag{6.73}$$

For $\sigma = 0$, and $\tau \geqslant 1/4$ Equation (6.72) becomes

$$\begin{aligned}[1-\tau p^2 a_m^{n+1}\,\delta_x^2]u_m^{n+1} \\ = 2\left[1-\left(\tau-\frac{1}{2}\right)p^2 a_m^n \delta_x^2\right] u_m^n - [1-\tau p^2\, a_m^{n-1}\,\delta_x^2]u_m^{n-1}\end{aligned} \tag{6.74}$$

which only requires the solution of a tridiagonal system for each n.

Equation (6.72) correct to $0(k^2+h^2)$ for arbitrary σ and τ may be written as

$$[1-\tau p^2 a_m^n\, Q_x^{-1}\,\delta_x^2](u_m^{n+1}-2u_m^n+u_m^{n-1}) = p^2 a_m^n\, Q_x^{-1}\,\delta_x^2\, u_m^n$$

which can be simplified to

$$[1+\sigma a_m^n\, \delta_x^2 (a_m^n)^{-1} - \tau p^2 a_m^n\, \delta_x^2](u_m^{n+1}-2u_m^n+u_m^{n-1}) = p^2 a_m^n\, \delta_x^2\, u_m^n \tag{6.75}$$

For $\sigma=\tau=1/12$, the difference scheme (6.75) has order of accuracy (h^4+k^4).

6.5.2 Two space dimensions

Here the linear hyperbolic equation under consideration is

$$\frac{\partial^2 u}{\partial t^2} = a(x, y, t)\frac{\partial^2 u}{\partial x^2} + b(x, y, t)\,\frac{\partial^2 u}{\partial y^2} \tag{6.76}$$

where $a(x, y, t) > 0$, $b(x, y, t) > 0$. The initial and boundary conditions are given by (6.32). The explicit difference replacement to (6.76) can be written as

$$\delta_t^2\, u_{l,m}^n = p^2 (a_{l,m}^n\, \delta_x^2 + b_{l,m}^n\, \delta_y^2)\, u_{l,m}^n \tag{6.77}$$

where $u_{l,m}^n$ is the approximate value of $u(x_l, y_m, t_n)$. The implicit difference formulas for (6.76) can be obtained from the equation

$$Q_t^{-1}\,\delta_t^2 u_{l,m}^n = p^2[a_{l,m}^n\, Q_x^{-1}\,\delta_x^2 u_{l,m}^n + b_{l,m}^n Q_y^{-1}\,\delta_y^2\, u_{l,m}^n] \tag{6.78}$$

where

$$Q_t = 1+\tau\delta_t^2, Q_x = 1+\sigma\delta_x^2 \text{ and } Q_y = 1+\sigma\delta_y^2 \tag{6.79}$$

The difference approximation represented by (6.78) has truncation error of $0(k^2+h^2)$ for arbitrary σ and τ. The values $\sigma=\tau=1/12$ increase the order of the truncation error to (k^4+h^4). Multiplying (6.78) by Q_t and simplifying, we obtain

$$\begin{aligned}\delta_t^2\,[1-\tau p^2 a_{l,m}^n Q_x^{-1}\,\delta_x^2][1-\tau p^2 b_{l,m}^n\, Q_y^{-1}\,\delta_y^2]\, u_{l,m}^n \\ = p^2 (a_{l\,m}^n Q_x^{-1}\,\delta_x^2 u_{l,m}^n + b_{l,m}^n\, Q_y^{-1}\,\delta_y^2 u_{l,m}^n)\end{aligned} \tag{6.80}$$

together with the initial conditions

$$u(x, y, 0) = 0$$

$$\frac{\partial u(x, y, 0)}{\partial t} = \sin^2 x \sin y$$

and the boundary conditions

$$u(\tfrac{1}{2}, y, t) = \sin^2 (\tfrac{1}{2}) \sin y \sin t$$
$$u(1, y, t) = \sin^2 (1) \sin y \sin t$$
$$u(x, \tfrac{1}{2}, t) = \sin (\tfrac{1}{2}) \sin^2 x \sin t$$
$$u(x, 1, t) = \sin (1) \sin^2 x \sin t$$

has been solved numerically with the various methods. The theoretical solution is given by

$$u(x, y, t) = \sin^2 x \sin y \sin t$$

The problem is solved first with $k = h/2 = 0.05$ and then with $k = h/2 = 0.025$. The maximum absolute errors are given in Table 6.3. For small value of the time step Ciment-Leventhal scheme produces best results, even though the difference scheme (6.86) with $\sigma = \tau = 1/12$ is of comparable accuracy. As expected other values of (σ, τ) give larger errors than these two schemes. However, when $p = 1$, the Mckee scheme and the Ciment-Leventhal scheme fail as this value violates the stability condition. The unconditionally stable formulas obtained from (6.86) give accurate and stable results.

TABLE 6.3 Maximum Absolute Error Values in the Solution of (6.76) Subject to (6.90) for $p=0.5$

(σ, τ)	*Time steps*	*h*	*Formula* (6.86)	*Ciment-Leventhal formula*
$\left(\frac{1}{12}, \frac{1}{12}\right)$	15	.1	5.8087-08	1.2597-08
	30	.05	3.8407-09	6.7558-10
	20	.1	7.4659-08	1.9266-08
	40	.05	5.3062-09	8.7666-10
$\left(\frac{1}{12}, \frac{1}{2}\right)$	15	.1	8.3094-06	
	30	.05	2.1234-06	
	20	.1	7.6924-06	
	40	.05	2.0135-06	
$\left(\frac{1}{12}, \frac{1}{8}\right)$	30	.05	2.1277-07	
	40	.05	2.0512-07	

6.6 LOCALLY ONE DIMENSIONAL (LOD) METHODS

The first step in LOD method is to replace the differential equation by a system of differential equations in one space variable and the second step consists in forming a difference equation system with intermediate functions. We shall examine the principle of the method by taking the wave equation in two and three space variables.

6.6.1 Two space dimensions

The wave equation in two space variables

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

can be replaced by two locally one-dimensional equations

$$\frac{1}{2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial y^2},\quad \frac{1}{2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2} \tag{6.91}$$

Applying (6.91) consecutively at time steps $k/2$, we obtain the following difference schemes

$$(1+\tau F(E_t))^{-1}F(E_t)u^{n-1/2}=\tfrac{1}{2}p^2(1+\sigma\delta_y^2)^{-1}\delta_y^2 u^{n-1/2}$$
$$(1+\tau F(E_t))^{-1}F(E_t)u^{n}=\tfrac{1}{2}p^2(1+\sigma\delta_x^2)^{-1}\delta_x^2 u^{n}$$
$$(1+\tau F(E_t))^{-1}F(E_t)u^{n+1/2}=\tfrac{1}{2}p^2(1+\sigma\delta_y^2)^{-1}\delta_y^2 u^{n+1/2} \tag{6.92}$$

where $F(E_t)=E_t^{-1/2}-2+E_t^{1/2}$, E_t is the shift operator, and τ, σ are arbitrary parameters. We denote the approximate value of $u(lh\ \ mh, nk)$ by u^n. Simplifying (6.92), we get

$$(1+a\delta_y^2)(u^{n-1}+u^n)=2[1+(a+\tfrac{1}{4}p^2)\delta_y^2]u^{n-1/2}$$
$$(1+a\delta_x^2)(u^{n-1/2}+u^{n+1/2})=2[1+(a+\tfrac{1}{4}p^2)\delta_x^2]u^{n}$$
$$(1+a\delta_y^2)(u^{n}+u^{n+1})=2[1+(a+\tfrac{1}{4}p^2)\delta_y^2]u^{n+1/2}$$

where

$$a=\sigma-\tfrac{1}{2}\tau p^2$$

By successively eliminating $u^{n-1/2}$, $u^{n+1/2}$ from these equations we obtain the equivalent difference scheme

$$(1+a\delta_x^2)(1+a\delta_y^2)(u^{n-1}-2u^n+u^{n+1})$$
$$=p^2[\delta_x^2+\delta_y^2+2\sigma\delta_x^2\delta_y^2+(\tfrac{1}{4}-\tau)p^2\delta_x^2\delta_y^2]u^n \tag{6.93}$$

The difference scheme (6.93) differs from (6.40) by the term

$$p^2(\tfrac{1}{4}-\tau)\delta_x^2\delta_y^2 u^n$$

TABLE 6.4 The Error Values in the Solution of the Wave Equation (6.31) Subject to the Initial and Boundary Conditions (6.56)
(All digits are to be multiplied by 10^{-2})

	Equivalent LOD and ADI methods			*LOD methods*			*High accuracy Fairweather-Mitchell method*
σ	1/8	0	1/2	0	1/12	1/12	
τ	1/4	1/4	1/4	1/2	1/2	1/6	
$t \backslash k$	$k = 0.06, \quad p = 0.66$						
0.6	− .0221	− .5287	− .1610	− .7698	− .4010	− .0812	− .0008
1.2	− .0831	+1.9795	+ .6039	−2.8792	−1.5025	+ .3046	+ .0031
1.8	+ .1542	−3.6495	−1.1181	+5.2935	−2.7742	− .5644	− .0058
2.4	− .1981	+4.6327	+1.4309	+6.6836	+3.5315	+ .7236	+ .0075
3.0	+ .1820	−4.1466	−1.3036	−5.9116	−3.1806	− .6617	− .0068
$t \backslash k$	$k = 0.075, \quad p = 0.825$						
0.6	− .0656	− .6014	− .2438	− .9678	− .6089	− .1225	+ .0001
1.2	+ .2498	+ .2855	+ .9279	+3.6728	+2.3140	+ .4663	− .0005
1.8	− .4651	−4.2293	−1.7243	−6.7670	−4.2815	− .8677	− .0004
2.4	+ .5977	+5.3705	+2.2073	+8.5212	+5.4359	+1.1138	− .2775
3.0	− .5477	−4.7928	−2.0051	−7.4635	−4.8493	−1.0178	− 60.9823
$t \backslash k$	$k = 0.1, \quad p = 1.1$						
0.6	− .2417	− .7522	− .4117	−1.3725	−1.0305	− .2061	+ .0034
1.2	+ .9448	+2.9354	+1.6081	+5.3451	+4.0175	+ .8056	− .0138
1.8	−1.7694	−5.4643	−3.0058	−9.8755	−7.4536	−1.5094	− .5955
2.4	+2.2745	+6.9367	+3.8465	+12.3508	+9.3993	+1.9362	− 715.3420
3.0	−2.0470	−6.1471	−3.4701	−10.5788	−8.2052	−1.9765	−8493.75

In vector and matrix notations it can be written as

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{A}\frac{\partial u}{\partial x} = 0 \qquad (6.102)$$

where $\mathbf{u} = [u_1 \quad u_2]^T$ and $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ a constant 2×2 matrix. It is easily verified that the eigenvalues of $\mathbf{A}$ are real and distinct and the corresponding eigenvectors are linearly independent. In general, we take $\mathbf{u} = [u_1\, u_2 \cdots u_N]^T$ an N-component vector and $\mathbf{A}$ a constant $N \times N$ matrix with real and distinct eigenvalues and N linearly independent eigenvectors. Further, we may determine a non-singular matrix $\mathbf{S}$ such that $\mathbf{A}$ can be transformed to diagonal form

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$$

where $\mathbf{D}$ is diagonal matrix with the diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_N$, the real eigenvalues of $\mathbf{A}$. Premultiplying (6.102) by $\mathbf{S}^{-1}$ we obtain

$$\frac{\partial}{\partial t}(\mathbf{S}^{-1}\mathbf{u}) - \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\frac{\partial}{\partial x}(\mathbf{S}^{-1}\mathbf{u}) = \mathbf{0}$$

which can be written as

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{D}\frac{\partial \mathbf{v}}{\partial x} = 0 \qquad (6.103)$$

where $\mathbf{v} = \mathbf{S}^{-1}\mathbf{u}$.

The equation (6.103) component-wise becomes

$$\frac{\partial v_i}{\partial t} - \lambda_i \frac{\partial v_i}{\partial x} = 0, \qquad i = 1, 2, \ldots, N \qquad (6.104)$$

Thus, we consider the difference schemes and related basic concepts with reference to a simple first order partial differential equation.

6.7.1 First order hyperbolic scalar equation

Let us consider the scalar equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad c \text{ real constant} \qquad (6.105)$$

together with appropriate initial and boundary conditions. We cover the specified domain by a rectangular network with spacing h and k in the x and t directions respectively. We denote by (m, n) the nodal points and also assume that u_m^n represents an approximation to the exact value $u(x_m, t_n)$. The three different approximations for (6.105) in which the time derivatives is

The exact solution of (6.105), satisfying (6.117) is given by

$$u(x, t) = \exp\,(i2\pi w(x-ct)) \tag{6.118}$$

where w and c are the wave number and the phase speed respectively. The equation (6.106ii) may be written as

$$u_m^{n+1} = (1-cp)u_m^n + cpu_{m-1}^n \tag{6.119}$$

Satisfying the initial condition (6.117), we seek the solution of (6.119) of the form

$$u_m^n = \xi^n \exp\,(i2\pi wmh) \tag{6.120}$$

Substituting (6.120) in (6.119) and simplifying we obtain

$$\xi = 1 - cp(1-\exp\,(-i2\pi wh)) = |\,\xi\,|\exp\,(-i2\pi wkc_1) \tag{6.121}$$

where

$$|\,\xi\,| = (1-4cp(1-cp)\sin^2 \pi wh)^{1/2} \tag{6.122}$$

and

$$c_1 = \frac{1}{2\pi wk}\tan^{-1}\frac{cp\sin(2\pi wh)}{1-2cp\sin^2(\pi wh)} \tag{6.123}$$

Further, when h is small and w is independent of h, the relations (6.122) and (6.123), respectively become

$$|\,\xi\,| = 1-\delta(2\pi w)^{2s}, \qquad s=1 \tag{6.124}$$

$$c_1 = c\left(1-\frac{1}{6}(1-cp)(1-2cp)(2\pi wh)^2+\ldots\right) \tag{6.125}$$

where δ is a some positive constant.
The solution (6.120) with (6.121), (6.122) and (6.123) may be written as

$$u_m^n = \exp\,(-2c(1-cp)\pi^2 w^2 ht_n)\exp\,(i2\pi w(x_m - c_1 t_n)) \tag{6.126}$$

The exact solution (6.118) oscillates periodically in time without damping. The numerical solution (6.126) displays damped oscillations, the amplitude being damped out as $\exp(-2c(1-cp)\pi^2w^2ht_n)$ and tends to zero as $n\to\infty$. Thus, the difference scheme (6.106ii) may not be suitable for integrating numerically (6.105) with $c > 0$ over long time interval. In view of (6.124) the difference scheme (6.106ii) is said to be *dissipative* of order $2s$, $s=1$, i.e., two. We now define the relative phase error as

$$E_p = \frac{P-P_E}{P_E} \tag{6.127}$$

where P is the phase of the numerical solution and P_E is the phase of the exact solution to (6.105). The relative phase error of (6.106ii), using (6.125) is written as

$$E_P = -\frac{1}{6}(1-cp)(1-2cp)(2\pi wh)^2+\ldots$$

which shows that the numerical solution lags behind the exact solution for $0 < cp < \frac{1}{2}$. From (6.125) we find that the phase speed c_1 depends on the wave number w and the difference scheme (6.106ii) is said to be *dispersive*. For $cp = 1$ we obtain

$$E_p = 0 \text{ and } |\xi| = 1$$

The solution of the differential equation (6.105) satisfies the difference equation (6.106ii).

Example 6.3 Solve the initial boundary value problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \qquad 0 \leqslant x \leqslant 1$$

$$u(0, t) = u(1, t)$$

$$u(x, 0) = g(x)$$

$$g(x) = \begin{cases} 0 & 0 \leqslant x \leqslant 0.25 \\ \dfrac{x - 0.25}{0.25} & 0.25 \leqslant x \leqslant 0.5 \\ \dfrac{0.75 - x}{0.25} & 0.5 \leqslant x \leqslant 0.75 \\ 0 & 0.75 \leqslant x \leqslant 1 \end{cases}$$

using the Lax-Wendroff method with $h = \frac{1}{8}$ and $p = \frac{1}{2}$.

The nodal points are

$$x_m = mh, \; t_n = nk \qquad 0 \leqslant m \leqslant 8, \; n = 0, 1, 2 \ldots$$

The Lax-Wendroff method for $p = \frac{1}{2}$ becomes

$$u_m^{n+1} = u_m^n - \frac{1}{2}(u_{m+1}^n - u_{m-1}^n) + \frac{1}{8}(u_{m+1}^n - 2u_m^n + u_{m-1}^n)$$

The initial and boundary conditions give

$$u_0^0 = 0 \qquad u_1^0 = 0 \qquad u_2^0 = 0$$

$$u_3^0 = \frac{1}{2} \qquad u_4^0 = 1 \qquad u_5^0 = \frac{1}{2}$$

$$u_6^0 = 0 \qquad u_7^0 = 0 \qquad u_8^0 = 0$$

$$u_0^n = u_8^n \quad \text{(periodic boundary conditions)}$$

We have

$$n = 0 \qquad u_m^1 = u_m^0 - \frac{1}{2}(u_{m+1}^0 - u_{m-1}^0) + \frac{1}{8}(u_{m+1}^0 - 2u_m^0 + u_{m-1}^0),$$

$$0 \leqslant m \leqslant 8$$

$$m = 0 \qquad u_0^1 = u_0^0 - \frac{1}{2}(u_1^0 - u_{-1}^0) + \frac{1}{8}(u_1^0 - 2u_0^0 + u_{-1}^0)$$

second order accuracy require modification. The Taylor series expansion in time gives

$$\mathbf{u}(x_m, t_{n+1}) = \mathbf{u}(x_m, t_n) + k\frac{\partial \mathbf{u}}{\partial t}(x_m, t_n) + \frac{1}{2}k^2 \frac{\partial^2 \mathbf{u}}{\partial t^2}(x_m, t_n) + \dots \tag{6.131}$$

Using the differential equation (6.128), we obtain

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t}(x_m, t_n) &= -\left(\mathbf{A}\frac{\partial \mathbf{u}}{\partial x}\right)_m^n \\ \frac{\partial^2 \mathbf{u}}{\partial t^2}(x_m, t_n) &= \left(\mathbf{A}\frac{\partial}{\partial x}\left(\mathbf{A}\frac{\partial \mathbf{u}}{\partial x}\right)\right)_m^n \end{aligned} \tag{6.132}$$

It is easy to verify the relation

$$\left(\frac{\partial}{\partial x}\left(\mathbf{A}\frac{\partial \mathbf{u}}{\partial x}\right)\right)_m^n = \frac{1}{h^2}[\Delta_x(\mathbf{A}_m^{n+1/2}\nabla_x) + \nabla_x(\mathbf{A}_m^{n+1/2}\Delta_x)]\mathbf{u}_m^n + 0(h^2)$$

The Lax-Wendroff scheme becomes

$$\begin{aligned} \mathbf{u}_m^{n+1} = \mathbf{u}_m^n - \frac{1}{2}p\mathbf{A}_m^{n+1/2}(\Delta_x + \nabla_x)\mathbf{u}_m^n + \frac{1}{4}p^2(\mathbf{A}_m^{n+1/2}\Delta_x\mathbf{A}_m^{n+1/2}\nabla_x \\ + \mathbf{A}_m^{n+1/2}\nabla_x\mathbf{A}_m^{n+1/2}\Delta_x)\mathbf{u}_m^n \end{aligned} \tag{6.133}$$

Similarly, we may show that the implicit Wendroff scheme (6.129 iv) becomes

$$\left[\mathbf{I} + \frac{1}{2}(\mathbf{I} + p\mathbf{A}_{m+1/2}^{n+1/2})\Delta_x\right]\mathbf{u}_m^{n+1} = \left[\mathbf{I} + \frac{1}{2}(\mathbf{I} - p\mathbf{A}_{m+1/2}^{n+1/2})\Delta_x\right]\mathbf{u}_m^n \tag{6.134}$$

6.7.3 Systems in conservation form

Here we assume that the matrix **A** in (6.128) is the Jacobian matrix of the derivatives of the vector function **f** with respect to the components of the unknown vector **u**. The equation (6.128) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0} \tag{6.135}$$

which may be written in the form

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = \mathbf{0} \tag{6.136}$$

where

$$\mathbf{A}(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

The equation (6.136) is said to be in conservation form. The Lax-Wendroff method may be derived as follows:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} = -\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}, \quad \frac{\partial^2 \mathbf{u}}{\partial t^2} &= \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{f}}{\partial t}\right) \\ &= -\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial t}\right) = \frac{\partial}{\partial x}\left(\mathbf{A}(\mathbf{u})\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\mathbf{A}(\mathbf{u})\frac{\partial \mathbf{f}}{\partial x}\right) \end{aligned} \tag{6.137}$$

Substituting (6.137) into (6.131) we get

$$\mathbf{u}(x_m, t_{n+1}) = \mathbf{u}(x_m, t_n) - k\left(\frac{\partial \mathbf{f}}{\partial x}\right)_m^n + \frac{1}{2}k^2\left(\frac{\partial}{\partial x}\left(\mathbf{A}(\mathbf{u})\frac{\partial \mathbf{f}}{\partial x}\right)\right)_m^n + O(k^3) \quad (6.138)$$

Replacing the derivatives by the difference expressions, the Lax-Wendroff method is given by

$$\mathbf{u}_m^{n+1} = \mathbf{u}_m^n - \frac{1}{2}p(\mathbf{f}_{m+1}^n - \mathbf{f}_{m-1}^n) + \frac{1}{2}p^2(\mathbf{A}_{m+1/2}^n(\mathbf{f}_{m+1}^n - \mathbf{f}_m^n)$$
$$- \mathbf{A}_{m-1/2}^n(\mathbf{f}_m^n - \mathbf{f}_{m-1}^n)) \quad (6.139)$$

where $\mathbf{A}_{m+1/2}^n = \mathbf{A}(\mathbf{u}_{m+1/2}^n)$ and $\mathbf{f}_m^n = \mathbf{f}(\mathbf{u}_m^n)$

The other difference schemes listed in (6.129) may also be written for (6.135). From computation view point the two-step methods are used to solve (6.136). We now list a few two-step methods.

Lax-Wendroff method

(i) $\bar{\mathbf{u}}_m^{n+1} = \frac{1}{2}(\mathbf{u}_{m+1}^n + \mathbf{u}_{m-1}^n) - \frac{p}{2}(\mathbf{f}_{m+1}^n - \mathbf{f}_m^n)$

(ii) $\mathbf{u}_m^{n+1} = \mathbf{u}_m^n - p(\bar{\mathbf{f}}_m^{n+1} - \bar{\mathbf{f}}_{m-1}^{n+1})$ (6.140)

Rubin-Burstein method 1

(i) $\bar{\mathbf{u}}_{m+1/2}^{n+1} = \frac{1}{2}(\mathbf{u}_{m+1}^n + \mathbf{u}_m^n) - \frac{1}{2}p(\mathbf{f}_{m+1}^n - \mathbf{f}_m^n)$

$\mathbf{u}_{m-1/2}^{n+1/2} = \frac{1}{2}(\mathbf{u}_m^n + \mathbf{u}_{m-1}^n) - \frac{1}{2}p(\mathbf{f}_m^n - \mathbf{f}_{m-1}^n)$

(ii) $\mathbf{u}_m^{n+1} = \mathbf{u}_m^n - p(\mathbf{f}_{m+1/2}^{n+1/2} - \mathbf{f}_{m-1/2}^{n+1/2})$ (6.141)

Rubin-Burstein method 2

(i) $\bar{\mathbf{u}}_{m+1/2}^{n+1/2} = \frac{1}{2}(\mathbf{u}_m^n + \mathbf{u}_{m+1}^n) - p(\mathbf{f}_{m+1}^n - \mathbf{f}_m^n)$

$\bar{\mathbf{u}}_{m-1/2}^{n+1} = \frac{1}{2}(\mathbf{u}_m^n + \mathbf{u}_{m-1}^n) - p(\mathbf{f}_m^n - \mathbf{f}_{m-1}^n)$

(ii) $\mathbf{u}_m^{n+1} = \mathbf{u}_m^n - \frac{p}{2}\left(\frac{1}{2}(\mathbf{f}_{m+1}^n - \mathbf{f}_{m-1}^n) + \bar{\mathbf{f}}_{m+1/2}^{n+1} - \bar{\mathbf{f}}_{m-1/2}^{n+1}\right)$ (6.142)

Gourlay-Morris method

(i) $\mathbf{u}_m^{n+1/2} = \frac{1}{2}(\mathbf{u}_{m+1/2}^n + \mathbf{u}_{m-1/2}^n) - \frac{p}{2}(\mathbf{f}_{m+1/2}^n - \mathbf{f}_{m-1/2}^n)$

(ii) $\mathbf{u}_m^{n+1} = \mathbf{u}_m^n - p(\mathbf{f}_{m+1/2}^{n+1/2} - \mathbf{f}_{m-1/2}^{n+1/2})$ (6.143) if **A** has K negative and $N-K$ positive eigenvalues, where

$$\mathbf{u}^{\mathrm{I}}=[u_1u_2\ldots u_K]^T$$
$$\mathbf{u}^{\mathrm{II}}=[u_{K+1}u_{K+2}\ldots u_N]^T$$

The differential equation (6.128) together with the initial condition or with one of the two initial and boundary conditions constitutes a well posed problem. For the scalar equation (6.105), the initial and boundary conditions (6.154)-(6.156) become

(i) $u(x, 0)=f(x), \quad 0 \leqslant x \leqslant 1$

(ii) $u(0, t)=g(t) \quad c>0$

or $u(1, t)=g(t) \quad c<0$ (6.157)

The numerical solution of the differential equation (6.105) with the initial condition (6.153) may be obtained using the procedure discussed in Subsection 5.4.1. We now illustrate the application of the explicit and implict difference schemes in solving (6.105) with appropriate initial and boundary conditions.

6.8.1 Initial boundary value problem

We solve the differential equation

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0 \qquad c>0 \tag{6.158}$$

with the initial boundary conditions (6.157),

$$u(x,0)=f(x), \qquad 0 \leqslant x \leqslant 1$$
$$u(0,t)=g(t), \qquad t \geqslant 0 \tag{6.159}$$

The nodal points are given by $x_m=mh$, $t_n=nk$, $m=0, 1, 2, \ldots, M$, $n=0, 1, 2, \ldots$ and $Mh=1$. The Lax-Wendroff scheme with initial and boundary conditions become

$$u_m^{n+1}=u_m^n-\frac{1}{2}cp(u_{m+1}^n-u_{m-1}^n)+\frac{1}{2}c^2p^2(u_{m+1}^n-2u_m^n+u_{m-1}^n) \tag{6.160}$$

$$u_m^o=f_m,\ u_o^{n+1}=g^{n+1} \quad 0 \leqslant m \leqslant M,\ n=0, 1, 2,\ldots \tag{6.161}$$

It is readily seen that numerically we need boundary condition at $x=1$ for solving (6.159)-(6.160). The additional numerical boundary condition may be determined by various means, keeping in view that this condition should not adversely affect the pure initial value stability condition, $0<cp \leqslant 1$. The following four numerical boundary conditions at $x=1$ may be used:

$$u_M^{n+1}=u_M^n-\frac{1}{2}pc(3u_M^n-4u_{M-1}^n+u_{M-2}^n)+\frac{1}{2}p^2c^2(u_M^n-2u_{M-1}^n+u_{M-2}^n) \tag{6.162}$$

$$u_M^{n+1} = u_M^n - pc(u_M^n - u_{M-1}^n) \tag{6.163}$$

$$u_M^{n+1} + u_{M-1}^{n+1} + pc(u_M^{n+1} - u_{M-1}^{n+1}) = u_M^n + u_{M-1}^n - pc(u_M^n - u_{M-1}^n)$$

or $$\left[1 - \frac{1}{2}(1-cp)\nabla_x\right] u_M^{n+1} = \left[1 - \frac{1}{2}(1+cp)\nabla_x\right] u_M^n \tag{6.164}$$

$$u_M^{n+1} = u_M^{n-1} - 2cp(u_M^n - u_{M-1}^n) - cp(u_M^{n+1} - 2u_M^n + u_M^{n-1}) \tag{6.165}$$

The condition (6.162) is given by

$$u(x_M, t_{n+1}) = u(x_M, t_n) + k\left(\frac{\partial u}{\partial t}\right)_M^n + \frac{1}{2}k^2\left(\frac{\partial^2 u}{\partial x^2}\right)_M^n + 0(k^3)$$

$$= u_M^n - cp\left(\frac{\partial u}{\partial x}\right)_M^n + \frac{1}{2}c^2p^2\left(\frac{\partial^2 u}{\partial x^2}\right)_M^n + 0(k^3)$$

or $$u_M^{n+1} = u_M^n - \frac{1}{2}cp(2\nabla_x + \nabla_x^2)u_M^n + \frac{1}{2}c^2p^2\nabla_x^2 u_M^n$$

The condition (6.163) is an aproximation of the differential equation using backward differences in time and space. The equation (6.164) is obtained by integrating (6.158) over the rectangular cell, $x_{M-1} \leqslant x \leqslant x_M, nk \leqslant t \leqslant (n+1)k$ (see Figure 6.5) with the help of the trapezoidal rule.

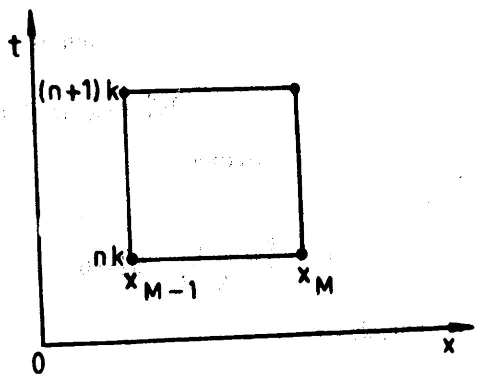

Fig. 6.5 Rectangular cell

We have

$$\int_{x_{M-1}}^{x_M} \int_{nk}^{(n+1)k} \left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) dx\, dt = 0$$

Integrating and then applying the trapezoidal rule, we get (6.164).

The condition (6.165) is obtained by using the Dufort-Frankel method for stabilizing the difference scheme.

Example 6.5 Solve the initial boundary value problem

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)=0 \qquad 0 \leqslant x \leqslant 1$$

$$u(x, 0)=x^2 \qquad 0 \leqslant x \leqslant 1$$

$$u(0, t)=0$$

using the Richtmyer method with $h=\frac{1}{3}$ and $p=\frac{1}{2}$.

The nodal points are

$$x_m=mh \qquad 0 \leqslant m \leqslant 3$$

$$t_n=nk \qquad n=0, 1, 2, \ldots$$

For $p=\frac{1}{2}$, the Richtmyer method may be written as

$$u_{m+1/2}^{n+1/2}=\frac{1}{2}(u_{m+1}^{n}+u_m^n)-\frac{1}{8}((u^2)_{m+1}^{n}-(u^2)_m^n)$$

$$u_m^{n+1}=u_m^n-\frac{1}{4}((u^2)_{m+1/2}^{n+1/2}-(u^2)_{m-1/2}^{n+1/2})$$

The initial boundary conditions become

$$u_m^0=m^2h^2, \qquad 0 \leqslant m \leqslant 3$$

or $\qquad u_0^0=0 \quad u_1^0=\frac{1}{9} \quad u_2^0=\frac{4}{9} \quad u_3^0=1$

$$u_0^{n+1}=0 \quad n=0, 1, 2, \ldots$$

We have

$$n=0,\ u_{m+1/2}^{1/2}=\frac{1}{2}(u_{m+1}^0+u_m^0)-\frac{1}{8}((u^2)_{m+1}^0-(u^2)_m^0)$$

$$m=0,\ u_{1/2}^{1/2}=\frac{1}{2}(u_1^0+u_0^0)-\frac{1}{8}((u_1^0)^2-(u_0^0)^2)=0.0540$$

$$m=1,\ u_{3/2}^{1/2}=\frac{1}{2}(u_2^0+u_1^0)-\frac{1}{8}((u_2^0)^2-(u_1^0)^2)=0.2546$$

$$m=2,\ u_{5/2}^{1/2}=\frac{1}{2}(u_3^0+u_2^0)-\frac{1}{8}((u_3^0)^2-(u_2^0)^2)=0.6219$$

$$m=1,\ u_1^1=u_1^0-\frac{1}{4}((u_{3/2}^{1/2})^2-(u_{1/2}^{1/2})^2)=0.0956$$

$$m=2,\ u_2^1=u_2^0-\frac{1}{4}((u_{5/2}^{1/2})^2-(u_{3/2}^{1/2})^2)=0.3640$$

For numerical boundary condition we use the method (6.163). We have for $p=\frac{1}{2}$

$$n=0,\quad u_3^1=u_3^0-\frac{1}{4}((u_3^0)^2-(u_2^0)^2)=0.7994.$$

The solution values obtained using analytic solution are

$$u_1^1=0.1003\quad u_2^1=0.3667\quad u_3^1=0.7621$$

6.8.2 Results from computation

The differential equation

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0 \tag{6.166}$$

has been solved subject to the following conditions

(a) $u(x, 0)=x^2, \quad 0 \leqslant x \leqslant 1$

$u(0, t)=0$

with the analytic solution

$$u(x, t)=[1+2xt-(1+4xt)^{1/2}]/2t^2$$

(b) $u(x, 0)=x, \quad 0 \leqslant x \leqslant 1$

$u(0, t)=0$

with the analytic solution

$$u(x, t)=x/(1+t)$$

(c) $u(x, 0)=\sqrt{x}, \quad 0 \leqslant x \leqslant 1$

$u(0, t)=0$

with the analytic solution

$$u(x, t)=[-t+(t^2+4x)^{1/2}]/2$$

(d) $u(x, 0)=(2x)^{1/2}, \quad 1 \leqslant x \leqslant 2$

$u(0, t)=(t^2+2)^{1/2}-t$

with the analytic solution

$$u(x, t)=(t^2+2x)^{1/2}-t$$

(e) $u(x, 0)=\begin{cases}1 & 0 \leqslant x < 0.1\\ 0 & x \geqslant 0.1\end{cases}$

$$u(0, t)=1 \qquad t>0 \tag{6.167}$$

The above problems have been solved with the Lax-Wendroff method using $p=1$ and 2, and $h=0.1$.

The results are given in Table 6.5.

TABLE 6.5 MAXIMUM ABSOLUTE ERRORS (ERROR × 10^{-4}) IN LAX-WENDROFF METHOD WITH $h=0.1$

Time steps	p	The differential equation $u_t+(\frac{1}{2}u^2)_x=0$ with initial boundary conditions			
		(a)	(b)	(c)	(d)
150	1	44	0.79(−07)	20.2	5.64
	2	20	0.38(−07)	11.1	*
300	1	45	0.12(−06)	15.1	1.98
	2	18	0.61(−07)	8.1	*

*unstable

The differential equation (6.166) with the discontinuous initial data (6.167e) has a discontinuous solution in which the discontinuity of the initial data is propagated into the field of solution along the line $x=0.1+t/2$. The solution obtained using the Lax-Wendroff method for $p=0.5$ and 1 and $h=0.01$ are given in Figure 6.6. The values of the solution after 50 time-steps are plotted for mesh points between $x=25\,ph$ and $x=25\,ph+15h$. The theoretical shock position occurs at $x=25\,ph+10h$.

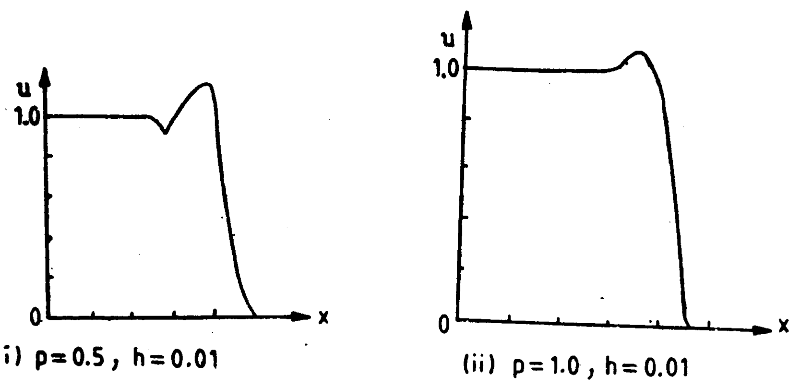

Fig. 6.6 Solution of $\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0$ with discontinuous data using Lax-Wendroff method

6.9 DIFFERENCE SCHEMES FOR SYSTEM OF EQUATIONS IN TWO SPACE VARIABLES

We now extend the methods studied in Sec. 6.7 to the case of two-space dimensional system

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{A}\frac{\partial \mathbf{u}}{\partial x}+\mathbf{B}\frac{\partial \mathbf{u}}{\partial y}=\mathbf{0} \tag{6.168}$$

where $\mathbf{u}$ is a vector function of the space coordinates x, y and the time t, and $\mathbf{A}$, $\mathbf{B}$ are symmetric matrices which may depend on x, y, t and $\mathbf{u}$. We wish to determine the solution of (6.168) in the region $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1, t \geqslant 0$, subject to the initial condition

$$\mathbf{u}(x, y, 0)=\mathbf{f}(x, y)$$

and the boundary conditions

$$\mathbf{u}(0, y, t),\ \mathbf{u}(1, y, t),\ \mathbf{u}(x, 0, t)$$

and $\mathbf{u}(x, 1, t)$ given for

$$0 \leqslant x, y \leqslant 1, \quad t \geqslant 0.$$

Further, we assume that there is no discontinuity in $\mathbf{u}$ between the initial and boundary conditions. The region $0 \leqslant x, y \leqslant 1, 0 \leqslant t \leqslant T$ is covered by a rectangular net parallel to the coordinate axes with h, k the space and time increments respectively.

The mesh ratio $p=k/h$ is assumed to be a constant. We denote by $\mathbf{u}^n_{l,m}=\mathbf{u}^n$ an approximate value of $u(lh, mh, nk)$.

Firstly, we assume that $\mathbf{A}$ and $\mathbf{B}$ are constant symmetric matrices but not necessarily commuting. The difference schemes listed in (6.129) may easily be written for (6.168)
We have

Diffusing scheme

(i) $\mathbf{u}^{n+1}=\mathbf{u}^n+\frac{1}{4}(\delta_x^2+\delta_y^2)\mathbf{u}^n-p(\mathbf{A}\mu_x\delta_x+\mathbf{B}\mu_y\delta_y)\mathbf{u}^n$

Lax-Wendroff scheme

(ii) $\mathbf{u}^{n+1}=\mathbf{u}^n-p(\mathbf{A}\mu_x\delta_x+\mathbf{B}\mu_y\delta_y)\mathbf{u}^n+\frac{1}{2}p^2(\mathbf{A}^2\delta_x^2+\mathbf{B}^2\delta_y^2$
$+(\mathbf{AB}+\mathbf{BA})\mu_x\delta_x\mu_y\delta_y)\mathbf{u}^n$

Leapfrog scheme

(iii) $\mathbf{u}^{n+1}=\mathbf{u}^{n-1}-2p(\mathbf{A}\mu_x\delta_x+\mathbf{B}\mu_y\delta_y)\mathbf{u}^n$

Gourlay-Mitchell scheme

(iv) $$\left(\mathbf{I}+\frac{1}{2}p\mathbf{B}\mu_y\delta_y\right)\left(\mathbf{I}+\frac{1}{2}p\mathbf{A}\mu_x\delta_x\right)\mathbf{u}^{n+1} = \left(\mathbf{I}-\frac{1}{2}p\mathbf{B}\mu_y\delta_y\right)\left(\mathbf{I}-\frac{1}{2}p\mathbf{A}\mu_x\delta_x\right)\mathbf{u}^{n}$$

Implicit scheme

(v) $$\mathbf{u}^{n+1}=\mathbf{u}^n-\frac{1}{2}p(\mathbf{A}\mu_x\delta_x+\mathbf{B}\mu_y\delta_y)(\mathbf{u}^n+\mathbf{u}^{n+1}) \tag{6.169}$$

If the matrices **A** and **B** are functions of x, y and t then the difference schemes in (6.169) need modifications. The Lax-Wendroff scheme may be written as

$$\mathbf{u}^{n+1}=\mathbf{u}^n-p(\mathbf{A}^{n+1/2}\mu_x\delta_x+\mathbf{B}^{n+1/2}\mu_y\delta_y)\mathbf{u}^n+\frac{1}{4}p^2(\mathbf{A}^{n+1/2}(\Delta_x A^{n+1/2}\nabla_x + \nabla_x\mathbf{A}^{n+1/2}\Delta_x))\mathbf{u}^n+\frac{1}{4}p^2\mathbf{B}^{n+1/2}(\Delta_y\mathbf{B}^{n+1/2}\nabla_y+\nabla_y\mathbf{B}^{n+1/2}\Delta_y)\mathbf{u}^n + \frac{1}{2}p^2(\mathbf{A}^{n+1/2}\mathbf{B}^{n+1/2}+\mathbf{B}^{n+1/2}\mathbf{A}^{n+1/2})\mu_x\delta_x\mu_y\delta_y\mathbf{u}^n \tag{6.170}$$

6.9.1 Stability analysis

We assume that

$$\mathbf{u}_{l,m}^{n}=\mathbf{u}_0\, e^{i(lh\beta_1+mh\gamma_1)} \tag{6.171}$$

where β_1 and γ_1 are arbitrary real numbers and $\mathbf{u}_0$, is a constant vector. The matrices **A** and **B** are taken to be constant. Substituting (6.171) into (6.169ii), we get

$$\mathbf{G}=(\mathbf{I}-p^2\mathbf{A}^2(1-\cos\beta_1 h)-p^2\mathbf{B}^2(1-\cos\gamma_1 h)-\frac{p^2}{2}(\mathbf{AB}+\mathbf{BA})\sin\beta_1 h\sin\gamma_1 h) + ip(\mathbf{A}\sin\beta_1 h+\mathbf{B}\sin\gamma_1 h) \tag{6.172}$$

Using the Lax-Richtmyer sufficient condition

$$\|\mathbf{G}^*\mathbf{G}\|\leqslant 1+0(k) \tag{6.173}$$

it may be verified that the Lax-Wendroff scheme (6.169ii) is stable if

$$0<p\leqslant\frac{1}{2\sqrt{2}\lambda_m}$$

where $\|\cdot\|$ denotes L_2 norm, $\mathbf{G}^*$ is the complex conjugate transpose of **G** and

$$|\lambda_m|=\max_{A,\,B}[\,|\lambda_A|\,,|\lambda_B|]$$

$$|\mathbf{A}-\lambda_A\mathbf{I}|=0\quad|\mathbf{B}-\lambda_B\mathbf{I}|=0$$